PHY 456 Lecture 7:
Addition of Angular Momenta (part II)

<u>Problem</u>: define some change of basis between the Hilbert spaces

$\{|j_1 m_1, j_2 m_2\rangle,\ m_1 = -j_1, \dots, j_1,\ m_2 = -j_2, \dots, j_2\}$ $(2j_1+1)(2j_2+1)$ - dimensional

$\{|j\, m\rangle,\ j = ?,\ m = ?\}$ Also must be $(2j_1+1)(2j_2+1)$-dimensional of eigenvectors of $\hat{J}^2, \hat{J}_z$ $\hat{J} = \hat{J}_1 + \hat{J}_2$.

For brevity, since j_1, j_2 are fixed, lets write our eigenvectors as $|m_1 m_2\rangle$.

Notice that $J_z |m_1, m_2\rangle = \hbar(m_1+m_2)|m_1, m_2\rangle$ since $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$

z-components add directly

which means that m_1, m_2 can range from

$-j_1-j_2, \dots, j_1+j_2$

implying that $\hat{J}_z$ ranges from $-\cancel{\hbar}(j_1+j_2)$ to $\cancel{\hbar}(j_1+j_2)$

Lets drop the $\hbar$.

So it must be that $\hat{J}^2$ has $j_{max} = j_1 + j_2$.

How to find j_{min}? Dimensionality.

→ Sum of all dimensions of Hilbert spaces must be $(2j_1+1)(2j_2+1)$.

Spin-$\frac{1}{2}$ example: $j_{max} = 1$, $j_{min} = 0$
$\left(\frac{1}{2} \otimes \frac{1}{2}\right)$ $= \frac{1}{2}+\frac{1}{2}$ $= \frac{1}{2}-\frac{1}{2}$

no loss of generality: "Guess" then verify later...

$$\left.\begin{aligned} j_{max} &= j_1+j_2 \\ j_{max}-1 &= j_1+j_2-1 \\ &\vdots \\ j_{min}+1 &= j_1-j_2+1 \\ j_{min} &= j_1-j_2 \end{aligned}\right\} \quad j = (\underbrace{j_1+j_2, \ldots, j_1-j_2}_{2j_2 \text{ values}})$$

where j_2 is the 'smaller spin'

The dimension of this new Hilbert space is $2(j_1+j_2-k)+1$

where k runs from $\underset{j_1+j_2}{0}$ to $\underset{j_1-j_2}{2j_2}$.

and the sum of total dimensions is

$$\sum_{k=0}^{2j_2}\left[2(j_1+j_2-k)+1\right] = 2\sum_{k=0}^{2j_2}\left(j_1+j_2+\frac{1}{2}\right) - 2\underbrace{\sum_{k=0}^{2j}k}_{\sum_{k=0}^{N} k = \frac{N(N+1)}{2}}$$

$$= (2j_2+1)(2(j_1+j_2)+1) - \frac{4j_2(2j_2+1)}{2}$$

$$= 4j_1j_2 + \cancel{4j_2^2} + \cancel{2j_2} + 2j_1 + 2j_2 + 1 - \cancel{4j_2^2} - \cancel{2j_2}$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1$$

$$= (2j_1+1)(2j_2+1)$$

so dimensionality matches.

Hence $j \in \{j_1+j_2,\ j_1+j_2-1,\ \dots,\ j_1-j_2+1,\ j_1-j_2\}$

2 examples: [1] $j_1 = 1$, $j_2 = \frac{1}{2}$.

Possible values $j = \frac{3}{2}, \frac{1}{2}$

j_1 3-dim $(-1, 0, 1)$ dim $j_{3/2} = 4 \left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

j_2 2-dim $(-\frac{1}{2}, \frac{1}{2})$ dim $j_{\frac{1}{2}} = 2 \left(\frac{1}{2}, -\frac{1}{2}\right)$

$3 \cdot 2 = 6$ $4 + 2 = 6$

dimensionality matches.

[2] $j_1 = 1$, $j_2 = 1$.

possible values $j = 2, 1, 0$.

j_1, j_2 both 3-dimensional

tot $= 9$

dim $j_2 = 5$ $(2, 1, 0, -1, -2)$

dim $j_1 = 3$ $(1, 0, -1)$

dim $j_0 = 1$ (0)

tot sum $= 9$.

dimensionality matches.

Next: how do we map

$\{|m_1, m_2\rangle\}$ - space to $\{|j, m\rangle\}$ - space?

We want to express $|j, m\rangle$ in terms of $|m_1 m_2\rangle$

with the previously found j's.

Observe the basis change

$$|j, m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1 j_2 m_2\rangle \underbrace{\langle j_1 m_1 j_2 m_2 | j\, m\rangle}_{\text{Clebsch-Gordon Coefficients.}}$$

For given j_1, j_2 we have the $|j\, m\rangle$ states

$j \rightarrow$ / $m \downarrow$	j_1+j_2	j_1+j_2-1	$---$	j_1-j_2
	$\|j_1+j_2, j_1+j_2\rangle$	$\times$		
	$\|j_1+j_2, j_1+j_2-1\rangle$	$\|j_1+j_2-1, j_1+j_2-1\rangle$		$\times$
	$\|j_1+j_2, j_1+j_2-2\rangle$	$\|j_1+j_2-1, j_1+j_2-2\rangle$	$--$	$\|j_1-j_2, j_1-j_2\rangle$
	$\vdots$	$\vdots$		$\vdots$
				$\|j_1-j_2, -j_1+j_2\rangle$
	$\|j_1+j_2, -j_1-j_2+1\rangle$	$\|j_1+j_2-1, -j_1-j_2+1\rangle$	$--$	
	$\|j_1+j_2, -j_1-j_2\rangle$	$\times$		$\times$

For top-right (1-1) state, can only be constructed out of $|j_1 m_1, j_2 m_2\rangle$ when m_1, m_2 are both max! There's no other state with $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z} = j_1 + j_2$.

We can move down the column by applying the total angular momentum lowering operator $\hat{J}_-$:

$$J_- |j_1+j_2, j_1+j_2\rangle = J_{1-}|j_1 j_1\rangle \otimes |j_2 j_2\rangle + |j_1 j_1\rangle \otimes J_{2-}|j_2 j_2\rangle$$

$$= \sqrt{j_1 + j_1 + j_2 + j_2}\ |j_1+j_2, j_1+j_2-1\rangle$$

$$= |j_1, j_1-1\rangle \otimes |j_2, j_2\rangle \sqrt{j_1+j_1} + |j_1 j_1\rangle \otimes |j_2, j_2-1\rangle \sqrt{j_2+j_2}$$

Hence $|j_1+j_2, j_1+j_2-1\rangle = |j_1, j_1-1, j_2, j_2\rangle \sqrt{\dfrac{j_1}{j_1+j_2}} + |j_1 j_1, j_2, j_2-1\rangle \sqrt{\dfrac{j_2}{j_1+j_2}}$

this is normalized

This is clean, and we can keep lowering the states with J_- in the j_1+j_2 column.

What about the $j = j_1 + j_2 - 1$ column? The top state has $m = j_1+j_2-1$ and is hence a linear combination of

$$|j_1\, j_1-1, j_2\, j_2\rangle \quad \text{and} \quad |j_1\, j_1, j_2\, j_2-1\rangle$$

which is should be orthogonal to

$$|j_1+j_2, j_1+j_2-1\rangle = |j_1\, j_1-1\, ;\, j_2\, j_2\rangle \sqrt{\frac{j_1}{j_1+j_2}} + |j_1\, j_1\, ;\, j_2\, j_2-1\rangle \sqrt{\frac{j_2}{j_1+j_2}}$$

are orthogonal
easy to check.

and thus

$$|j_1+j_2-1, j_1+j_2-1\rangle = |j_1\, j_1\, ;\, j_2\, j_2-1\rangle \sqrt{\frac{j_1}{j_1+j_2}} - |j_1\, j_1-1\, ,\, j_2\, j_2\rangle \sqrt{\frac{j_2}{j_1+j_2}}$$

and both states are normalized to 1.

Can similarly apply J_- to obtain all states in that column...

Move to third column $j = j_1+j_2-2$:

→ top state: $|j_1+j_2-2\, ,\, j_1+j_2-2\rangle$

but there is an $m=j_1+j_2-2$ state in both col. 1 and 2;

can get $m = j_1+j_2-2$ by adding three states

$$\left.\begin{array}{l} |j_1\, j_1-1;\, j_2\, j_2-1\rangle \\ |j_1\, j_1\, ;\, j_2\, j_2-2\rangle \\ |j_1\, j_1-2\, ;\, j_2\, j_2\rangle \end{array}\right\}$$

under 2 conditions:
- all orthogonal
- all normalized

. . .

We could do all of them, but this takes way too much time.

In general:

$$|j\,m\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} |j_1 m_1, j_2 m_2\rangle \underbrace{\langle j_1 m_1, j_2 m_2 | j m\rangle}_{\text{CG coefficients}}$$

Properties

1) $\langle j_1 m_1, j_2 m_2 | j m\rangle \neq 0$ iff $j = \{j_1+j_2, j_1+j_2-1, \dots, j_1-j_2\}$
iff $m = m_1 + m_2$

2) All $\langle j_1 m_1, j_2 m_2 | j m\rangle \in \mathbb{R}$

3) $\langle j_1 m_1, j_2 m_2 | j m\rangle > 0.$

4) $\langle j_1 m_1, j_2 m_2 | j m\rangle = (-1)^{j_1+j_2-j} \langle j_1 -m_1, j_2 -m_2 | j -m\rangle$

5) Matrix of CG is unitary

eg for $\frac{1}{2} \otimes \frac{1}{2} = 1 + 0$

$$\begin{array}{ccccc} |jm\rangle & & & & |m_1 m_2\rangle \\ \begin{pmatrix} |1\,1\rangle \\ |1\,0\rangle \\ |1\,-1\rangle \\ |0\,0\rangle \end{pmatrix} & = & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} & & \begin{pmatrix} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{pmatrix} \end{array}$$

Atom in Electric Field

$$\Delta\hat{H} = -e\,\vec{r}\cdot\vec{E} \qquad (-\vec{d}\cdot\vec{E})$$

Addition to $\hat{H}_{atom}$ due to $\vec{E}$-field to see effect on levels "Stark effect". Need to know various matrix elements

$i = 1, 2, 3$

$$\langle \underbrace{n\,\ell\,m}_{\text{with } \vec{E}} | \Delta\hat{H} | \underbrace{n'\,\ell' m'}_{\text{without } \vec{E}} \rangle \sim \langle n\,\ell\,m | r^i | n'\ell' m' \rangle$$

Now, $\{r^i\} = \{r_z,\ r_x + i r_y,\ r_x - i r_y\}$

$$= r\,\{\cos\theta,\ \sin\theta e^{i\varphi},\ \sin\theta e^{-i\varphi}\}$$

$$\sim r\,\{\underbrace{Y_1^0,\ Y_1^1,\ Y_1^{-1}}\}$$

Spherical Harmonics with $\ell = 1$

This means that the states $\{r^i|\ell m\rangle\} = \{Y_1^0|\ell m\rangle, Y_1^1|\ell m\rangle, Y_1^{-1}|\ell m\rangle\}$ transform under rotations exactly as the states $|\ell_1 m_1; \ell_2 m_2\rangle$

$\ell_1 = 1,\ m_1 = -1, 0, 1$

but then the matrix element of ΔH is

$$\Delta H \sim \langle \ell' m' | 1, 0, \ell, m \rangle, \langle \ell' m' | 1, 1, \ell m \rangle, \langle \ell' m' | 1, -1, \ell m \rangle$$

$$= \text{the CG coefficients for } \ell \otimes 1 = (\ell+1) \oplus (\ell-1) \; !$$

so ℓ' can only be ± 1

A particular case of the Wigner-Eckart theorem.

→ $\vec{r}$ is an "irreducible vector operator" ($\ell = 1 \leftrightarrow i = 1, 2, 3$)

→ $r_i r_j - \frac{1}{3} r^2 \delta_{ij}$ = traceless symmetric 3x3 matrix with 5 elements

$\ell = 2$ irreducible tensor